COM701 - Numerical Mathematics and Computing

Chapter 2 - Taylor Series and Polynomial Approximations

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AY - 2025/2026

Outline

1. Taylor Series Fundamentals
2. Complete Horner's Algorithm
3. Taylor's Theorem in Terms of $(x - c)$
4. Mean-Value Theorem
5. Taylor's Theorem in Terms of h
6. Alternating Series
7. Hands-On Numerical Experiments

Outline

Taylor Series Fundamentals

Taylor series is a powerful mathematical tool used to **approximate functions** as **infinite sums of terms** calculated from the values of their **derivatives at a single point**.

→ *It is widely used in numerical analysis, physics, and engineering for function approximation, solving differential equations, and analyzing the behavior of functions near specific points.*

Taylor Series Fundamentals

Familiar Taylor Series The following Taylor series are commonly used in numerical mathemathics:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (|x| < \infty) \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (|x| < \infty) \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad (|x| < \infty) \tag{3}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k \quad (|x| < 1) \tag{4}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \quad (-1 < x < 1) \tag{5}$$

Taylor Series Fundamentals

Example 1: Approximating ln(1.1)

Use five terms of the Taylor series (5) to approximate $\ln(1.1)$.

Solution

To approximate $\ln(1.1)$, we set $x = 0.1$ in the Taylor series expansion of $\ln(1+x)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \quad (-1 < x \leq 1)$$

$$\begin{aligned}
\ln(1.1) &= \ln(1+0.1) \\
&\approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \frac{(0.1)^5}{5} \\
&\approx 0.1 - 0.005 + 0.0003333 - 0.000025 + 0.000002 \\
&\approx 0.0953103
\end{aligned}$$

Note: The actual value of $\ln(1.1)$ is approximately 0.09531018.
Conclusion: The approximation using five terms of the Taylor series is quite accurate, with an error of approximately $0.00000012 = 1.2 \times 10^{-7}$. *This value is accurate to six decimal places.*
Final remark: However, such accurate results are not always guaranteed when using Taylor series.

Taylor Series Fundamentals

Example 2: Approximating e^8

Use six terms of the Taylor series (1) to approximate e^8.

Solution

To approximate e^8, we can use the Taylor series expansion of e^x ~~*around $x = 0$*~~:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (|x| < \infty)$$

$$\begin{aligned} e^8 &\approx 1 + 8 + \frac{8^2}{2!} + \frac{8^3}{3!} + \frac{8^4}{4!} + \frac{8^5}{5!} \\ &\approx 1 + 8 + \frac{64}{2} + \frac{512}{6} + \frac{4096}{24} + \frac{32768}{120} \\ &\approx 1 + 8 + 32 + 85.3333 + 170.6667 + 273.0667 \\ &\approx 570.06667 \end{aligned}$$

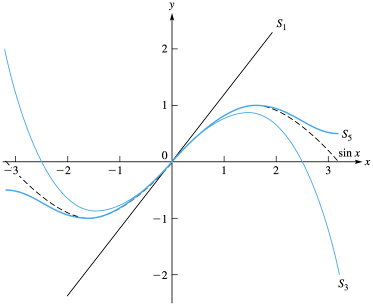
Note: The actual value of e^8 is approximately 2980.957987.
Conclusion: The approximation using the first six terms of the Taylor series gives a value of approximately 570.06667, which is significantly lower than the actual value of 2980.957987. *This indicates that the Taylor series converges slowly for larger values of x.*
Final remark: It is apparent that many terms will be needed to compute e^8 with reasonable precision.

Taylor Series Fundamentals

Example 3: Graphing the Taylor Series sin(x) for 3 partial sums

Plot the graphs of the Taylor series (2) for sin(x) *using the three first partial sums.*

Solution



$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (|x| < \infty)$$

$$S_1(x) = x$$

$$S_3(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

$$S_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}$$

Conclusion: The graph illustrates how the Taylor series approximations of sin(x) improve as more terms are included in the partial sums. The first partial sum $S_1(x)$ provides a rough approximation, while $S_3(x)$ and $S_5(x)$ offer increasingly accurate representations of the sin(x) function over a wider range of x values.

Taylor Series Fundamentals

All of the series previously illustrated are examples of the following general series:

Theorem 1: Formula Taylor Series for f about c

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$
$$\approx \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \quad (6)$$

where $f^{(k)}(c)$ is the k^{th} derivative of f evaluated at the point c.

→ *The point c is called the* **expansion point** *or* **point of expansion** *or* **center** *of the Taylor series. When $c = 0$, the series is also known as a* **Maclaurin series**.

Taylor Series Fundamentals

Example 4: Taylor series of a polynomial

What is the Taylor series of the function

$$f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5$$

at the point $c = 2$?

Step 1: Compute derivatives at $x = 2$

To compute the coefficients in the series, we need $f^{(k)}(2)$ for $k \geq 0$.

k	$f^{(k)}(x)$	$f^{(k)}(2)$
0	$f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5$	207
1	$f'(x) = 15x^4 - 8x^3 + 45x^2 + 26x - 12$	396
2	$f''(x) = 60x^3 - 24x^2 + 90x + 26$	590
3	$f'''(x) = 180x^2 - 48x + 90$	714
4	$f^{(4)}(x) = 360x - 48$	672
5	$f^{(5)}(x) = 360$	360
6	$f^{(6)}(x) = 0$	0 for all $k \geq 6$

Taylor Series Fundamentals

Example 4: Taylor series of a polynomial

Step 2: Construct the Taylor series

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots \approx \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

The Taylor series of $f(x)$ about $c = 2$ is:

$$f(x) = 207 + 396(x-2) + \frac{590}{2!}(x-2)^2 + \frac{714}{3!}(x-2)^3 + \frac{672}{4!}(x-2)^4 + \frac{360}{5!}(x-2)^5$$

$$= 207 + 396(x-2) + 295(x-2)^2 + 119(x-2)^3 + 28(x-2)^4 + 6(x-2)^5$$

Note: Since $f(x)$ is a polynomial of degree 5, the Taylor series expansion terminates after the fifth derivative; all higher-order derivatives are zero.

Remark: In this case, "$\approx$" may be replaced by "$=$". Expanding and simplifying the Taylor series recovers the original polynomial $f(x)$. *$\rightarrow$Taylor's Theorem (coming next) justifies this conclusion directly.*

Conclusion: The Taylor series expansion provides an **exact representation** of $f(x)$ around $c = 2$.

Final Remark: In general, Taylor series converge more rapidly for values of x close to the expansion point c. *For larger values of $|x-c|$, more terms may be needed to achieve a good approximation.*

Taylor Series Fundamentals

Key Takeaways

- *Taylor series* provide a way to **approximate complex functions** using **polynomials**, which are **easier to compute and analyze**.
- The *accuracy of the approximation* depends on the **number of terms used** and the **point around which the series is expanded**.
- **Taylor series converge more rapidly** for values of x **close to** the *expansion point*, and **may require many terms** for **larger values** of x.
- **Caution:** It is important to consider the **radius of convergence** and the **behavior of the function** when using **Taylor series for approximations**.
- **Taylor's Theorem** provides a formal framework for understanding the **accuracy and convergence** of Taylor series approximations.

Outline

Complete Horner's Algorithm

An application of Horner's algorithm is that of finding the Taylor expansion of a polynomial about any point.

Let $p(x)$ be a given polynomial of degree n with coefficients a_k as in the equation (1) in Chapter 1, and suppose that we desire the coefficients c_k in the equation

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \\ &= c_n (x-r)^n + c_{n-1}(x-r)^{n-1} + \ldots + c_1 (x-r) + c_0 \end{aligned} \tag{7}$$

Taylor's Theorem asserts that $c_k = p^{(k)}(r)/k!$, but we seek a more **efficient algorithm**. Notice that $p(r) = c_0$, so **this coefficient is obtained by applying Horner's algorithm** to the polynomial p with the point r. *The algorithm also yields the polynomial:*

$$q(x) = \frac{p(x) - p(r)}{(x-r)} = c_n (x-r)^{n-1} + c_{n-1}(x-r)^{n-2} + \ldots + c_2 (x-r) + c_1 \tag{8}$$

Complete Horner's Algorithm

$$q(x) = \frac{p(x) - p(r)}{(x - r)} = c_n(x - r)^{n-1} + c_{n-1}(x - r)^{n-2} + \ldots + c_2(x - r) + c_1 \tag{8}$$

- **This shows that the second coefficient, c_1, can be obtained by applying Horner's algorithm to the polynomial $q(x)$ at the point r, since $c_1 = q(r)$.**
- **Note that the first application of Horner's algorithm does not yield $q(x)$ in the form shown above, but rather as a sum of powers of x.** (See Equations (4) and (5) in Chapter 1)
  - *This process is repeated iteratively to compute all coefficients c_k in the expansion in powers of $(x - r)$.*
  - *Each coefficient c_k can be obtained by successively applying Horner's algorithm to the resulting polynomial divided by $(x - r)^k$.*

Complete Horner's Algorithm

The *Complete Horner's Algorithm* is an **extension** of the **classic Horner's method**. **It evaluates a polynomial and all of its derivatives at a given point x, using an in-place computation**[1].

Given a polynomial:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

The input list of coefficients $a = [a_0, a_1, \ldots, a_n]$ is overwritten such that for $k = 0, 1, \ldots, n$:

- $a[0] = p(x)$ (value of the polynomial)
- $a[1] = p'(x)$ (first derivative)
- $a[2] = \frac{p''(x)}{2!}$ (second derivative)

  $\vdots$
- $a[k] = \frac{p^{(k)}(x)}{k!}$ (k^{th} derivative)

[1] In-place computation updates the existing data structure instead of creating a new one.

Complete Horner's Algorithm

Below is the Python implementation of the *Complete Horner's Algorithm*, which modifies the coefficient list in-place.

```python
def complete_horner(a, x):
    """
    Inputs:
    a: list of coefficients [a0, a1, ..., an] of P(x)
    x: value at which to evaluate p and its derivatives
    Output:
    a: modified in place, where
       a[0] = p(x), a[1] = p'(x), ..., a[k] = p^(k)(x)/k!
    """
    n = len(a) - 1
    for k in range(1, n + 1):
        for j in range(n, k - 1, -1):
            a[j - 1] += x * a[j]
    return a  # a[0], a[1], ..., a[n] now contain results
```

Click here to download and test the code

Complete Horner's Algorithm

Example 5: Taylor Expansion of a Polynomial using Complete Horner's Algorithm

Using the complete Horner's algorithm, find the Taylor expansion of the polynomial $p(x) = x^4 - 4x^3 + 7x^2 - 5x + 2$ *about the point* $r = 3$.

Solution

The work can be arranged as follows:

3)	1	−4	7	−5	2
	↓	3	−3	12	21
	1	−1	4	7	**23**
	↓	3	6	30	
	1	2	10	**37**	
	↓	3	15		
	1	5	**25**		
	↓	3			
	1	**8**			
	1				

The calculation shows that

$$p(x) = (x-3)^4 + \mathbf{8}(x-3)^3 + \mathbf{25}(x-3)^2 + \mathbf{37}(x-3) + \mathbf{23}$$

Complete Horner's Algorithm

Exercise: Taylor Expansion of a Degree-6 Polynomial using Complete Horner's Algorithm

Using the complete Horner algorithm, find the Taylor expansion about the point $r = 2$ *of the polynomial* $p(x) = 2x^6 - 9x^5 + 15x^4 - 14x^3 + 6x^2 + x - 5.$

Solution

Complete Horner's Algorithm

Exercise: Taylor Expansion of a Degree-6 Polynomial using Complete Horner's Algorithm

Using the complete Horner algorithm, find the Taylor expansion about the point $r = 2$ of the polynomial $p(x) = 2x^6 - 9x^5 + 15x^4 - 14x^3 + 6x^2 + x - 5$.

Solution

2)	2	−9	15	−14	6	1	−5
	↓	4	−10	10	−8	−4	−6
	2	−5	5	−4	−2	−3	**-11**
	↓	4	−2	6	4	4	
	2	−1	3	2	2	**1**	
	↓	4	6	18	40		
	2	3	9	20	**42**		
	↓	4	14	46			
	2	7	23	**66**			
	↓	4	22				
	2	11	**45**				
	↓	4					
	2	**15**					
	2						

$$p(x) = \mathbf{2}(x-2)^6 + \mathbf{15}(x-2)^5 + \mathbf{45}(x-2)^4 + \mathbf{66}(x-2)^3 + \mathbf{42}(x-2)^2 + \mathbf{1}(x-2) \mathbf{-11}$$

Outline

Taylor's Theorem in Terms of $(x - c)$

Theorem 2: Taylor Theorem for $f(x)$

If the function f possesses continuous derivatives of orders $0, 1, 2, \ldots, (n+1)$ in a closed interval $I = [a, b]$, then for any c and x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1} \tag{9}$$

where the error term E_{n+1} can be given in the form

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

Here ξ is a point that **lies** between c and x and depends on both. e.g., if $c < x$, then $\xi \in (c, x)$; if $x < c$, then $\xi \in (x, c)$.

Taylor's Theorem in Terms of $(x - c)$

- In practical computations with **Taylor series**, it is usually necessary to *truncate* the series because it is **not possible to carry out an infinite number of additions**.
- A series is said to be *truncated* if we **ignore all terms after a certain point**.

Thus, if we truncate the exponential Series (see Equation (1)) after seven terms, the result is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

This no longer represents e^x except when $x = 0$. But the truncated series should approximate e^x .

$\rightarrow$ **Here is where we need Taylor's Theorem. With its help, we can assess the difference between a function f and its truncated Taylor series.**

Taylor's Theorem in Terms of $(x - c)$

Example 6: Taylor series for $f(x) = \ln(1+x)$

Derive the formal Taylor series for $f(x) = \ln(1+x)$ at $c = 0$, and determine the range of positive x for which the series represent the function.

Solution

We need $f^{(k)}(x)$ and $f^{(k)}(0)$ for $k \geq 0$. the general formula for the n^{th} derivative of $f(x) = \ln(1+x)$ is $f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}, \quad n \geq 1$

The derivatives are

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$f(x) = \ln(1+x)$	$f(0) = 0$
1	$f'(x) = (1+x)^{-1}$	$f'(0) = 1$
2	$f''(x) = -(1+x)^{-2}$	$f''(0) = -1$
3	$f'''(x) = 2(1+x)^{-3}$	$f'''(0) = 2$
4	$f^{(4)}(x) = -6(1+x)^{-4}$	$f^{(4)}(0) = -6$
$\vdots$	$\vdots$	$\vdots$
k	$f^{(k)}(x) = (-1)^{k-1}(k-1)!(1+x)^{-k}$	$f^{(k)}(0) = (-1)^{k-1}(k-1)!$

Taylor's Theorem in Terms of $(x - c)$

Example 6: Taylor series for $f(x) = \ln(1 + x)$

Solution

Hence by Taylor's Theorem seen in Equation (9),

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k + E_{n+1}, \text{ where } E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - c)^{n+1} \text{ and } \xi \in (c, x) \text{ or } (x, c)$$

the Taylor series for $f(x) = \ln(1 + x)$ at $c = 0$ is given by

$$\ln(1 + x) = \sum_{k=1}^{n}(-1)^{k-1}\frac{(k-1)!}{k!}x^k + \frac{(-1)^n n!(1+\xi)^{-n-1}}{(n+1)!}x^{n+1} = \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k} + \frac{(-1)^n}{n+1}(1+\xi)^{-n-1}x^{n+1}$$

$$= \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k} + \underbrace{\frac{(-1)^n}{n+1}\left(\frac{x}{1+\xi}\right)^{n+1}}_{E_{n+1}}, \text{ where } \xi \in (0, x) \text{ or } (x, 0) \tag{10}$$

- **The above series represents $\ln(1 + x)$ when the error term $E_{n+1} \to 0$ as $n \to \infty$.**
- For **the case of $0 \leq x \leq 1$**: $0 < \xi < x$ (because zero is the point of expansion); thus, $0 \leq \frac{x}{1+\xi} \leq 1$ **that ensures $E_{n+1} \to 0$ and the series converges**.
- For $x > 1$: terms do not vanish, so the series diverges.
- The series also converges for $-1 < x < 0$, but not for $x \leq -1$.

Taylor's Theorem in Terms of $(x - c)$

Example 7: Taylor series for e^x and its convergence

Derive the Taylor series for e^x at $c = 0$, and prove that it converges to e^x by using Taylor's Theorem.

Solution

The function $f(x) = e^x$ has the property that all its derivatives are equal to e^x. Thus, $f^{(k)}(c) = f^{(k)}(0) = e^0 = 1$ for all $k \geq 0$. Therefore, from Equation (9), the **Taylor series for e^x at $c = 0$** is given by

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + E_{n+1} = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{\xi}}{(n+1)!} x^{n+1}, \quad \text{where } \xi \in (0, x) \text{ or } (x, 0) \tag{11}$$

Now let us consider all the values of x in some symmetric interval around the origin, for example, $-s \leq x \leq s$. Then $|x| \leq s$, $|\xi| \leq s$, and $e^{\xi} \leq e^s$. Hence, the remainder term satisfies this inequality:

$$\lim_{n \to \infty} |E_{n+1}| = \lim_{n \to \infty} \left| \frac{e^{\xi}}{(n+1)!} x^{n+1} \right| \leq \lim_{n \to \infty} \frac{e^s}{(n+1)!} s^{n+1} = 0$$

Thus, if we take the limit as $n \to \infty$ on both sides of the Equation (11), we obtain

$$e^x = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!} + \lim_{n \to \infty} E_{n+1} = \sum_{k=0}^{\infty} \frac{x^k}{k!} + 0 = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Conclusion: The Taylor series for e^x at $c = 0$ converges to e^x for all real numbers x.

Taylor's Theorem in Terms of $(x - c)$

Key Takeaways

- **Taylor's Theorem (in terms of $(x - c)$):** A function can be approximated around a point c by a polynomial using its derivatives, plus an error term:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1}, \quad E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}, \quad \xi \in (c, x) \text{ or } (x, c)$$

- **Assumptions:** $f, f', \ldots, f^{(n+1)}$ are continuous on $I = [a, b]$.
- **Remainder:** valid if $f^{(n+1)}$ exists on (a, b).
- **Evaluation point:** ξ depends on x and lies between c and x.
- **Structure of E_{n+1}:** resembles the next term of the series, but $f^{(n+1)}$ is evaluated at ξ instead of c.

Outline

Mean-Value Theorem

The special case $n = 0$ in Taylor's Theorem is known as the *Mean-Value Theorem*. It is usually stated, however, in a somewhat more precise form.

Theorem 3: Mean-Value Theorem

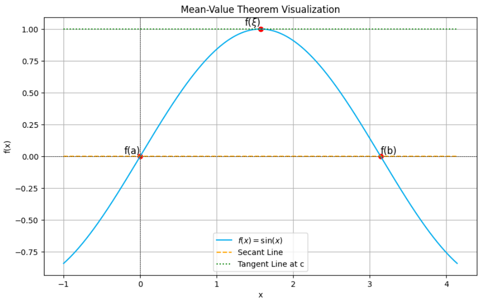
If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one point ξ in (a, b) such that

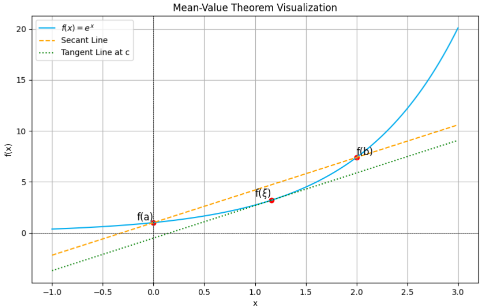
$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \tag{12}$$

- This means that there is **at least one point ξ in the interval (a, b)** where the **instantaneous rate of change (the derivative) of the function equals the average rate of change over the entire interval**.
- The **right-hand** side could be used as an **approximation for $f'(x)$ at any x within the interval (a, b)**. *The approximation of derivatives is discussed more fully in Chapter 7.*

Mean-Value Theorem

Geometric Interpretation: The *Mean-Value Theorem* states that there is at least one point on the curve of $f(x)$ between $x = a$ and $x = b$ where the tangent line is parallel to the secant line connecting the points $(a, f(a))$ and $(b, f(b))$.





Outline

Taylor's Theorem in Terms of h

Other forms of *Taylor's Theorem* are often useful. These can be obtained from the basic Formula (9) by changing the variables.

Theorem 4: Taylor's Theorem for $f(x+h)$

If the function f possesses continuous derivatives of orders $0, 1, 2, \ldots, (n+1)$ in a closed interval $I = [a, b]$, then for any x in I,

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1} \tag{13}$$

where h is any value such that $x + h$ is in I and where

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}$$

for some ξ between x and $x + h$.

Taylor's Theorem in Terms of h

- Theorem 4 (see Equation (13)) is obtained from Theorem 2 (see Equation(9)) by replacing x by $x+h$ and replacing c by x.
- Notice that because h can be positive or negative, the requirement on ξ means:
  - $x < \xi < x + h$ if $h > 0$, or
  - $x + h < \xi < x$ if $h < 0$.
- The *error term* E_{n+1} depends on h in two ways:
  - First, h^{n+1} is explicitly present;
  - Second, the point ξ generally depends on h.
- As h converges to zero, E_{n+1} converges to zero with essentially the same rapidity with which h^{n+1} converges to zero. For large n, this is quite rapid. To express this qualitative fact, we write

$$E_{n+1} = O(h^{n+1}) \text{ as } h \to 0$$

This is called *big O notation* and it is shorthand for the inequality $|E_{n+1}| \leq C|h|^{n+1}$ where C is a constant.

Taylor's Theorem in Terms of h

Example 8: $\sqrt{1+h}$ expansion

Expand $\sqrt{1+h}$ in powers of h. Then compute $\sqrt{1.00001}$ and $\sqrt{0.99999}$.

Solution

Step 1: Compute derivatives of $f(x) = \sqrt{x}$:

$$f'(x) = \frac{1}{2}x^{-1/2}, \quad f''(x) = -\frac{1}{4}x^{-3/2}, \quad f'''(x) = \frac{3}{8}x^{-5/2}, \ldots$$

Step 2: Write the Taylor series expansion around $x = 1$:

$$f(1+h) = f(1) + f'(1)h + \frac{f''(1)}{2}h^2 + \frac{f'''(\xi)}{6}h^3, \quad 1 < \xi < 1+h$$

Step 3: Substitute derivatives:

$$\sqrt{1+h} = 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \frac{1}{16}h^3\xi^{-5/2}, \quad 1 < \xi < 1+h \tag{14}$$

Step 4: Approximate $\sqrt{1.00001}$ with $h = 10^{-5}$ using the three first derivatives:

$$\sqrt{1.00001} \approx 1 + 0.5 \cdot 10^{-5} - 0.125 \cdot 10^{-10} = 1.00000\,49999\,87500$$

Remark: The function $\sqrt{x}$ has derivatives of all orders for $x > 0$, so the series can be extended further if needed.

Taylor's Theorem in Terms of h

Example 8 (continued)

Solution (continued)

Step 5: Approximate $\sqrt{0.99999}$ by substituting $-h$ for h using the three first derivatives:

$$\sqrt{1-h} = 1 - \frac{1}{2}h - \frac{1}{8}h^2 - \frac{1}{16}h^3\xi^{-5/2}, \quad 1-h < \xi < 1 \tag{15}$$

Step 6: Numerical approximation:

$$\sqrt{0.99999} \approx 0.99999\,49999\,87500$$

Step 7: Bound the remainder: Since $1 < \xi < 1+h$, it means that $\xi^{-5/2}$ is bounded between $1^{-5/2} = 1$ and $(1+h)^{-5/2} \approx 1 - \frac{5}{2}h$. Thus, the **absolute error** does not exceed

$$\left|\frac{1}{16}h^3\xi^{-5/2}\right| < \frac{1}{16}10^{-15} = 0.00000\,00000\,00000\,0625$$

Conclusion: Both approximations are correct to all 15 decimal places shown.

Taylor's Theorem in Terms of h

Key Takeaways

- **Taylor's Theorem (in terms of h):** A function can be approximated around a point x by a polynomial using its derivatives, plus an error term:

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1}, \quad E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}, \quad \xi \in (x, x+h) \text{ or } (x+h, x)$$

- **Assumptions:** $f, f', \ldots, f^{(n+1)}$ are continuous on $I = [a, b]$.
- **Remainder:** valid if $f^{(n+1)}$ exists on (a, b).
- **Evaluation point:** ξ depends on h and lies between x and $x+h$.
- **Structure of E_{n+1}:** resembles the next term of the series, but $f^{(n+1)}$ is evaluated at ξ instead of x.
- **Big O notation:** $E_{n+1} = O(h^{n+1})$ as $h \to 0$, meaning there exists a constant C such that $|E_{n+1}| \leq C|h|^{n+1}$ for sufficiently small h.

Outline

Alternating Series

Another useful result from calculus helps in **establishing the convergence of a series** and in **estimating the error when truncating a series**.

From this result, we get an important principle for alternating series:

Principle

If the magnitudes[a] of the terms in an alternating series decrease to zero, then the error made by truncating the series is no greater than the magnitude of the first omitted term.

[a] The magnitude of a term refers to its absolute value. Here, the sequence of magnitudes (a_n) is decreasing: $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots > 0$.

Note: This principle applies only to *alternating series*—*series in which successive terms are alternately* **positive** *and* **negative**.

Alternating Series

Theorem 5: Alternating Series Theorem

Suppose $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots > 0$ for all n and $\lim_{n\to\infty} a_n = 0$. Then the alternating series

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

converges; that is,

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k = \lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k-1}a_k = \lim_{n\to\infty} S_n = S,$$

where S is the sum and S_n is the n^{th} partial sum.

Moreover, for all n,

$$|S - S_n| \leq a_{n+1}.$$

Alternating Series

Example 9: Using the sine series to compute sin(1)

Compute $\sin(1)$ *using its Taylor series with an error less than* $\frac{1}{2} \times 10^{-6}$. *How many terms are needed?*

Solution

From the Taylor series of $\sin(x)$:

$$\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + (-1)^{n-1}\frac{1}{(2n-1)!} = S_n$$

By the **Alternating Series Theorem**, if we truncate after the n-th term, the error satisfies

$$|S - S_n| \leq \frac{1}{(2n+1)!}.$$

We require

$$\frac{1}{(2n+1)!} < \frac{1}{2} \times 10^{-6}.$$

Checking factorial values, $(2 \times 4 + 1)! = 9! = 362{,}880$, so $\frac{1}{9!} \approx 2.76 \times 10^{-6} > \frac{1}{2} \times 10^{-6}$. (**fails**)
Checking factorial values, $(2 \times 5 + 1)! = 11! \approx 3.99 \times 10^{7}$, so $\frac{1}{11!} \approx 2.5 \times 10^{-8} < \frac{1}{2} \times 10^{-6}$. (**succeeds**)
Conclusion: $n = 5$ terms suffice to achieve the desired accuracy.

Alternating Series

Example 10: Using the logarithmic series to compute ln(2)

Compute ln(2) *using its Taylor series with an error less than* $\frac{1}{2} \times 10^{-6}$. *How many terms are needed?*

Solution

The Taylor series of $\ln(1+x)$ at $x = 1$ is:

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{n} = S_n$$

By the **Alternating Series Theorem**, the error in truncating after n terms satisfies

$$|S - S_n| \leq \frac{1}{n+1}.$$

To meet the desired accuracy:

$$\frac{1}{n+1} < \frac{1}{2} \times 10^{-6} \quad \Rightarrow \quad n > 2{,}000{,}000.$$

Conclusion: More than two million terms are needed, so this method is impractical for computing ln(2) directly.

Alternating Series

Key Takeaways

- **Alternating Series Theorem:** If $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots > 0$ and $\lim_{n\to\infty} a_n = 0$, then the alternating series
$$a_1 - a_2 + a_3 - a_4 + \cdots$$
converges to S, and for all n,
$$|S - S_n| \leq a_{n+1}.$$
- This theorem applies only to *alternating series*—series in which successive terms are alternately positive and negative.
- The truncation error is bounded by the magnitude of the first omitted term.
- **Example 9:** To compute sin(1) with an error less than $\frac{1}{2} \times 10^{-6}$, $n = 5$ terms suffice.
- **Example 10:** To compute ln(2) with an error less than $\frac{1}{2} \times 10^{-6}$, more than two million terms are needed, making this method impractical.

Outline

1. Taylor Series Fundamentals
2. Complete Horner's Algorithm
3. Taylor's Theorem in Terms of $(x - c)$
4. Mean-Value Theorem
5. Taylor's Theorem in Terms of h
6. Alternating Series
7. Hands-On Numerical Experiments

Hands-On Numerical Experiments

- To gain a better understanding of Taylor series and their convergence properties, it is beneficial to perform numerical experiments.
- Download the Jupyter notebook (ipynb file) and follow the instructions provided within the notebook to explore and more deeply understand the concepts discussed in this chapter.

End of Chapter 2